

# ON METRICS ON 2-ORBIFOLDS ALL OF WHOSE GEODESICS ARE CLOSED

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**ABSTRACT.** We show that the geodesic period spectrum of a Riemannian 2-orbifold all of whose geodesics are closed depends, up to a constant, only on its orbifold topology and compute it. In the manifold case we recover the fact proved by Gromoll, Grove and Pries that all prime geodesics have the same length. In the appendix we partly strengthen our result in terms of conjugacy of contact forms and explain how to deduce rigidity on the real projective plane based on a systolic inequality due to Pu.

## 1. INTRODUCTION

Riemannian manifolds all of whose geodesics are closed have been studied since the beginning of the 20th century, when the first nontrivial examples were constructed by Tannery and Zoll. The famous book of Besse [Be78] still describes the state of knowledge of the subject to a large extent. Some notable exceptions are concerned with relations between the geodesic period spectrum of a Riemannian manifold all of whose geodesics are closed, henceforth called Besse manifold, and its topology. For instance a conjecture of Berger, stating that on a simply connected Besse manifold all prime geodesics have the same length, was proved by Gromoll and Grove for 2-spheres [GG81] and recently by Radeschi and Wilking for all topological spheres of dimension at least 4 [RW15]. Apart from spheres, which admit many Besse metrics, i.e. Riemannian metrics all of whose geodesics are closed, the only known Besse manifolds are the other compact rank one symmetric spaces. Moreover, it was shown by Pries that the conclusion of Berger's conjecture also holds for the real projective plane [Pr09], i.e. that all prime geodesics of a Besse metric have the same length.

Only little is known in the more general setting of Riemannian orbifolds. We define a *Besse metric* on an orbifold as a Riemannian orbifold metric all of whose orbifold geodesics are closed and a *Besse orbifold* as an orbifold endowed with a Besse metric (cf. Section 2.1). On Besse orbifolds new phenomena occur that are not present in the manifold case. For instance, Berger's conjecture does not hold for spindle orbifolds [GUW09], which admit many Besse metrics (cf. Section 2.2). However, it turns out that there is still a relation between the geodesic period spectrum of a Besse 2-orbifold and its topology. In fact, we generalize the results of Gromoll, Grove and Pries mentioned above in a unifying approach to the setting of Riemannian 2-orbifolds. We prove the following result.

**Theorem.** *The geodesic period spectrum of a Besse 2-orbifold is determined up to a constant by its orbifold topology. In the manifold case all prime geodesics have the same length.*

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1991 *Mathematics Subject Classification.* 53C22, 57R18, 55R55.

The geodesic period spectrum of a Besse 2-orbifold is by definition the set of lengths of all unoriented prime geodesics counted with multiplicity. In the manifold case the proofs by Gromoll, Grove and Pries hinge on the existence of at least three simple closed geodesics, i.e. closed geodesics without self-intersections. Using a connectedness argument they moreover show that all prime geodesics are simple. This observation combined with the Blaschke conjecture for  $S^2$  proved by Green [Gr63] shows that a Besse metric on the real projective plane has constant curvature [Pr09]. Our proof is independent of the existence of simple closed geodesics. In the appendix we show how rigidity on the real projective plane can be deduced from our result based on a systolic inequality due to Pu [Pu52]. The latter observation was explained to us by A. Abbondandolo. In the appendix we moreover partially strengthen our result by showing that the geodesic flow on the unit tangent bundle of a Besse 2-orbifold with isolated singularities can be conjugated by a contactomorphism to a standard geodesic flow.

The paper is structured as follows. After reviewing some preliminaries and examples, we will see in Section 2.4 that a 2-orbifold admits a Besse metric if and only if its orbifold Euler characteristic is positive (cf. Proposition 2.4). This is the case if and only if it is either a quotient of  $S^2$  or bad, i.e. not covered by a manifold. Quotients of  $S^2$  by finite subgroups of  $O(3)$  inherit canonical Besse metrics from  $S^2$ . Besse metrics on bad 2-orbifolds, can be constructed as quotients of  $S^3$  by almost-free isometric  $S^1$ -actions [GUW09] and  $\mathbb{Z}_2$ -quotients thereof. Other constructions via metrics of revolution are given in [Be78] and will be discussed in Section 2.2. The space of oriented prime geodesics on a Besse 2-orbifold has a natural orbifold structure (cf. Section 2.7). We show that its orbifold topology only depends on the orbifold topology of the given Besse orbifold in the following way.

**Theorem.** *Let  $\mathcal{O}$  be a Besse 2-orbifold and let  $\mathcal{O}_g$  be the corresponding orbifold of oriented geodesics. The following cases can occur.*

- (i)  $\mathcal{O} \cong S^2/G$  and  $\mathcal{O}_g \cong S^2/G^\times$  as orbifolds where  $G < O(3)$  is a finite subgroup and  $G^\times = \{\det(g)g | g \in G\} < SO(3)$ .
- (ii)  $\mathcal{O} \cong S^2(p, q)$  and  $\mathcal{O}_g \cong S^2((p+q)/\kappa, (p+q)/\kappa)$  with  $\kappa$  being 1 or 2 depending on whether  $p+q$  is odd or even.
- (iii)  $\mathcal{O} \cong D^2(; p, q)$  and  $\mathcal{O}_g \cong S^2(2, 2, (p+q)/\kappa)$  with  $\kappa$  being 1 or 2 depending on whether  $p+q$  is odd or even.

For explanations on the notations we refer to Section 2. Based on the above list we will be able to show that the geodesic period spectrum also only depends on the orbifold topology of a given Besse 2-orbifold. The proof of the above list relies on the following ideas. The unit tangent bundle  $M = T^1\mathcal{O}$  of a Besse 2-orbifold  $\mathcal{O}$  with isolated singularities is a manifold and the geodesic flow on it is periodic due to a result of Epstein [Ep72]. We obtain two transversal Seifert fiberings on  $M$ , one from the geodesic flow and another one from the projection  $T^1\mathcal{O} \rightarrow \mathcal{O}$ . Properties of these Seifert fiberings and their interplay imply the claim in many cases. For Besse 2-orbifolds with codimension one singularities the result is obtained by considering the orientable double cover. In this case geometric arguments come into play. An explicit list of all 2-orbifolds admitting a Besse metric together with their orbifolds of prime geodesics and their geodesic period spectra can be found in the Appendix, Table 1

Independently Frauenfelder and Suhr used similar methods to show a Hamiltonian version of the result of Gromoll and Grove on an assumption analogous to the existence of three simple closed geodesics [FS16]. As in the Riemannian case our proof shows that this assumption can

be avoided (cf. [FLS16]). Note that in general our result is a Riemannian phenomenon that cannot be seen from the Hamiltonian point of view. For instance, the real projective plane and the teardrop  $S^2(3)$  have the same unit tangent bundle (cf. Lemma 3.1 and Section 3.2), but the period spectra of Besse metrics on them differ (cf. appendix, Table 1).

*Acknowledgements.* The author would like to thank Alexander Lytchak for drawing his attention to the subject. He is grateful to Alberto Abbondandolo for explaining to him how this paper's result combined with work by Pu implies rigidity on the real projective plane (cf. appendix).

## 2. PRELIMINARIES

**2.1. Orbifolds.** For a definition of a (smooth) orbifold we refer to [BH99] or [Da11]. A Riemannian orbifold can be defined as follows.

**Definition 2.1.** An  $n$ -dimensional Riemannian orbifold  $\mathcal{O}$  is a length space such that for each point  $x \in \mathcal{O}$ , there exists a neighborhood  $U$  of  $x$  in  $\mathcal{O}$ , an  $n$ -dimensional Riemannian manifold  $M$  and a finite group  $\Gamma$  acting by isometries on  $M$  such that  $U$  and  $M/\Gamma$  are isometric.

A *length space* is a metric space in which the distance of any two points can be realized as the infimum of the lengths of all rectifiable paths connecting these points [BBI01]. Behind the above definition lies the fact that an isometric action of a finite group on a simply connected Riemannian manifold can be recovered from the corresponding metric quotient [La16]. In particular, a Riemannian orbifold in the above sense admits a smooth orbifold structure and a compatible Riemannian structure that in turn induces the metric structure. For a point  $x$  on a Riemannian orbifold, the isotropy group of a preimage of  $x$  in a Riemannian manifold chart is uniquely determined up to conjugation. Its conjugacy class in  $O(n)$  is denoted as  $\Gamma_x$  and is called the *local group* of  $\mathcal{O}$  at  $x$ . Riemannian orbifolds are stratified by manifolds. The  $k$ -dimensional stratum consists of those points  $x \in \mathcal{O}$  for which  $\dim(\text{Fix}(\Gamma_x)) = k$ .

The underlying topological space  $|\mathcal{O}|$  of a 2-orbifold  $\mathcal{O}$  is a manifold with boundary. The orbifold  $\mathcal{O}$  is orientable, if and only if  $|\mathcal{O}|$  is an orientable surface without boundary. A 2-orbifold can have three types of singularities. Isolated singularities in the interior of  $|\mathcal{O}|$  whose local groups are cyclic and orientation preserving, mirror singularities on the boundary of  $|\mathcal{O}|$  whose local groups are generated by a reflection, and corner-reflector singularities on the boundary of  $|\mathcal{O}|$  whose local groups are dihedral groups generated by two distinct reflections. The closure of the 1-dimensional stratum is the boundary of  $|\mathcal{O}|$  and consists of the mirror- and the corner-reflector singularities. We denote a 2-orbifold  $\mathcal{O}$  with  $l$  isolated singularities in the interior of  $|\mathcal{O}|$  (whose local group are) of order  $n_1, \dots, n_l$  and  $k$  isolated corner-reflector singularities on the boundary of  $|\mathcal{O}|$  whose local groups are dihedral of order  $2m_1, \dots, 2m_k$  by  $\mathcal{O} = |\mathcal{O}|(n_1, \dots, n_l; m_1, \dots, m_k)$ . If the boundary of  $|\mathcal{O}|$  is empty we simply write  $\mathcal{O} = |\mathcal{O}|(n_1, \dots, n_l)$ .

We are interested in (orbifold) geodesics in the following sense

**Definition 2.2.** An (*orbifold*) *geodesic* on a Riemannian orbifold is a continuous path that can locally be lifted to a geodesic in a Riemannian manifold chart.

In particular, we are interested in Riemannian orbifold metrics all of whose geodesics are closed, henceforth called *Besse metrics*. We call a Riemannian orbifold whose metric is Besse a

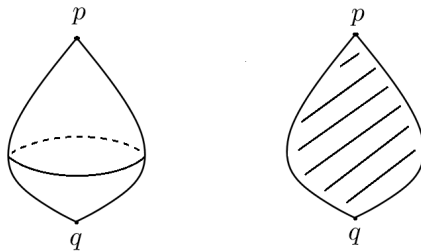


FIGURE 1. Left: A  $(p, q)$ -spindle orbifold  $S^2(p, q)$ . Right: A  $(p, q)$ -half-spindle orbifold  $D^2(; p, q)$ . Note that we do not demand  $p \neq q$  or  $p, q \neq 1$ .  $(p, 1)$ -spindle orbifold are also known as teardrops. The orbifolds in the picture are bad if and only if  $p \neq q$ .

*Besse orbifold.* Since a Besse orbifold is geodesically complete, it is complete by the Hopf-Rinow theorem [BBI01, Thm. 2.5.28].

Let us summarize basic properties of geodesics on a complete Riemannian 2-orbifold  $\mathcal{O}$ . In the regular part geodesics behave like ordinary geodesics in a Riemannian manifold. A geodesic hitting an isolated singularity is either reflected or goes straight through it depending on whether the order of the singularity is even or odd. Suppose the 1-dimensional stratum of  $\mathcal{O}$  is non-empty. Then its metric double  $\hat{\mathcal{O}}$  is a Riemannian orbifold with isolated singularities that admits an isometric involution with corresponding quotient  $\mathcal{O}$ . A geodesic hitting the topological boundary of  $\mathcal{O}$  continues as the projection of its continuation in  $\hat{\mathcal{O}}$  to  $\mathcal{O}$ . In particular, a geodesic hitting the 1-dimensional stratum is reflected according to the usual reflection law. A geodesic that remains in the closure of the 1-dimensional stratum for a positive time, stays there forever.

We need the following concept.

**Definition 2.3.** A *covering orbifold* or *orbi-cover* of a Riemannian orbifold  $\mathcal{O}$  is a Riemannian orbifold  $\mathcal{O}'$  together with a surjective map  $\varphi : \mathcal{O}' \rightarrow \mathcal{O}$  such that each point  $x \in \mathcal{O}$  has a neighborhood  $U$  isometric to some  $M/G$  for which each connected component  $U_i$  of  $\varphi^{-1}(U)$  is isometric to  $M/G_i$  for some subgroup  $G_i < G$  such that the isometries respect the projections.

Note that a finite orbi-cover of a Besse orbifold is itself Besse. In particular, the metric double cover of a Besse 2-orbifolds with mirror singularities is a Besse 2-orbifold with only isolated singularities. An orbifold is called *good* (or *developable*) if it is covered by a manifold. Otherwise it is called *bad*. The only bad 2-orbifolds are depicted in Figure 1 (cf. [Sc03, Thm. 2.5]).

For orbifolds an Euler characteristic can be defined that is multiplicative under coverings and coincides with the usual Euler characteristic in the manifold case [Da11]. A 2-orbifold has positive Euler characteristic if and only if it is either bad or *spherical*, i.e. a quotient of  $S^2$  by a finite subgroup of  $O(3)$ . All spherical 2-orbifolds are listed in Table 1 for  $p = q$  (cf. [Da11]). A detailed description of the corresponding finite subgroups of  $O(3)$  can for instance be found in [LL77]. Spherical 2-orbifolds inherit a standard Besse metric from  $S^2$ . In Proposition 2.4 we will see that a 2-orbifold admits a Besse metric if and only if its orbifold Euler characteristic is positive. The 2-orbifolds with positive Euler characteristic are listed in Table 1.

**2.2. Besse metrics on 2-orbifolds.** In [GUW09] a Besse  $(p, q)$ -spindle orbifold (cf. Figure 1) is constructed for integers  $p$  and  $q$  with  $\gcd(p, q) = 1$  and  $1 < p < q$  as follows. Consider the action

$$\begin{aligned} S^1 \times S^3 &\longrightarrow S^3 \\ (z, (z_1, z_2)) &\longmapsto (z^p z_1, z^q z_2). \end{aligned}$$

The quotient  $S^3/S^1$  is a Besse  $(p, q)$ -spindle orbifold. The geodesics on  $S^3/S^1$  are precisely the projections of *horizontal* geodesics on  $S^3$ , i.e. geodesics that are orthogonal to the  $S^1$ -orbits. In [GUW09, Thm. 3.6] the lengths of the geodesics' trajectories on the quotient  $S^3/S^1$  are computed. The difference to our result in the case that  $p + q$  is odd is due to the fact that the length of a geodesic's trajectory may differ from its period. In fact, closed geodesics that hit a singularity of even order pass their trajectory twice during a single period.

Another construction of Besse metrics on  $(p, q)$ -spindle orbifolds for arbitrary  $p$  and  $q$  is similar to the construction non-standard Besse (Zoll) metrics on  $S^2$  due to Tannery and Zoll (cf. [Be78]). Let  $h : [-1, 1] \rightarrow (-\frac{p+q}{2}, \frac{p+q}{2})$  be a smooth odd function with  $h(1) = \frac{p-q}{2} = -h(-1)$  and let a Riemannian metric on  $X = (0, \pi) \times ([0, 2\pi]/0 \sim 2\pi) \ni (\theta, \phi)$  be defined by

$$g = \left( \frac{p+q}{2} + h(\cos(\theta)) \right)^2 d\theta^2 + \sin^2(\theta) d\phi^2.$$

Then the metric completion of  $(X, g)$  is a Besse  $(p, q)$ -spindle orbifold (cf. [Be78, Thm. 4.13] and note that  $p$  and  $q$  are defined differently therein). Since the metric is invariant under a reflection in  $\phi \in ([0, 2\pi]/0 \sim 2\pi)$ , it descends to a Besse metric on the corresponding quotient. Therefore every bad 2-orbifold admits a Besse metric. In particular, we have by now seen Besse metrics on each 2-orbifold with positive Euler characteristic.

**2.3. Almost-free circle actions on 3-manifolds and Seifert fiber spaces.** Suppose we have a smooth, almost-free (i.e. isotropy groups are finite)  $S^1$ -action on an orientable closed 3-manifold  $M$ . Then the orbits are circles and define a decomposition of  $M$  into so-called *fibers*. If some element of  $S^1$  fixes a point on a fiber, then it fixes the fiber pointwise. A fiber is called *exceptional* of order  $k \geq 2$  if its isotropy subgroup of  $S^1$  has order  $k$ . Since  $S^1$  is compact, there exists an  $S^1$ -invariant Riemannian metric on  $M$ . The metric quotient  $M/S^1$  is an orientable Riemannian 2-orbifold with isolated singularities and (the orders of) the exceptional fibers correspond to (the orders of) the singularities of  $M/S^1$  (cf. [LT10]). The manifold  $M$  together with a chosen orientation and its decomposition into fibers defines a *Seifert fiber space* of type  $o_1$  (or  $Oo$  in Seifert's original notation [Sei33]) (cf. [JN83, Ch. 2]). As such it is uniquely determined by a finite number of numerical invariants up to orientation- and fiber-preserving diffeomorphism [JN83, Thm. 1.5]. Two sets of invariants determine the same Seifert fiber space, if and only if they are related as described in [JN83, Thm. 1.5] (cf. Section 4.3). Forgetting about the orientation of  $M$  and allowing general fiber preserving diffeomorphisms amounts to enlarging the equivalence relation on the set of numerical invariants (cf. [JN83, Cor. 1.7]). In [Ray68, (6.1)] it is shown that two  $S^1$ -actions in question on  $M$  define the same Seifert fiber space up to orientation, if and only if there exists a diffeomorphism  $h$  of  $M$  and an automorphism  $a$  of  $S^1$  such that for all  $m \in M$  and  $g \in S^1$  the relation  $h(gm) = a(g)h(m)$  holds. In this case it can in fact be shown that this automorphism can be chosen to be the identity (cf. [JN83, pp. 12-13]). A specific diffeomorphism that conjugates the  $S^1$ -actions occurring in this paper to their inverse actions is described in Section 2.7 (cf. the map  $i : T^1\mathcal{O} \rightarrow T^1\mathcal{O}$ ). Hence, the

classification of smooth, almost free  $S^1$ -actions on  $M$  up to conjugation by diffeomorphisms coincides with the classification of Seifert fiberings on  $M$  of type  $o_1$  up to orientation.

Note that the classifications of Seifert fibered spaces in the topological and the smooth category coincide [JN83, Sei33]. A standard fibered solid torus is a solid torus  $T = D^2 \times S^1$  fibered by the orbits of the almost-free  $S^1$ -action  $e^{it}(re^{it_0}, e^{it_1}) = (re^{it}e^{it_0}, e^{ikt}e^{it_1})$  for some positive integer  $k$ . Every smooth Seifert fibering on a solid torus  $T$  is fiber preservingly diffeomorphic to precisely one of the standard fibered solid tori. Suppose we have two fibered solid tori  $T_1$  and  $T_2$  and a fiber-preserving diffeomorphism  $\psi : \partial T_1 \rightarrow \partial T_2$ . Then we obtain a new Seifert fiber space  $T_1 \cup_\psi T_2$  by gluing together  $T_1$  and  $T_2$  via  $\psi$ . Let  $m_1$  be a meridian of  $T_1$ , i.e. an embedded loop in  $\partial T_1$  that is null-homotopic in  $T_1$  and that generates a maximal subgroup of  $H_1(\partial T_1)$ . Then the fiber-homeomorphism (and hence fiber-diffeomorphism) type of  $T_1 \cup_\psi T_2$  is completely determined by the homology class of  $\psi(m_1)$  in  $H_1(\partial T_2)$  (cf. [Br93, Thm. 1.3.4.]).

**2.4. Lens spaces.** Recall that for coprime integers  $p, q \neq 0$  the  $L(p, q)$  lens space is defined as a quotient of  $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$  by the free  $\mathbb{Z}_p$ -action on  $S^3$  given by  $e^{2\pi i/p}(z_1, z_2) = (e^{2\pi i/p}z_1, e^{2\pi iq/p}z_2)$ . Also recall that two lens spaces  $L(p, q)$  and  $L(p', q')$  are diffeomorphic, if and only if  $p = \pm p'$  and  $q \equiv \pm q'^{\pm 1} \pmod p$  (cf. [JN83, pp. 28-29]). An alternative construction of lens spaces works as follows. If, in the situation of the preceding section, we choose meridians  $m_i$  on  $T_i$  and longitudes  $l_i$  on  $T_i$ , i.e. embedded loops in  $\partial T_i$  that generate maximal subgroups of  $H_1(T_i)$ , and if we have  $\psi(m_1) \sim sm_2 + rl_2$  in  $H_1(\partial T_2)$ , then the space  $T_1 \cup_\psi T_2$  obtained by gluing together  $T_1$  and  $T_2$  via  $\psi$  is a  $L(r, -s)$  lens space (cf. [Br93, Thm. 1.3.4.] and [JN83, Thm. 4.3]). In particular, it is a  $L(p, 1)$  lens space if and only if  $r = \pm p$  and  $s \equiv \pm 1 \pmod p$ .

**2.5. Wadsley's result.** In [Wa75] Wadsley proves the following result (cf. [Be78, Thm. 7.12]).

**Theorem.** *If the orbits of a flow on a Riemannian manifold are periodic geodesics parametrized by arc-length, then the flow itself is periodic, so that the orbits have a common period.*

**2.6. 2-orbifolds that admit Besse metrics.** The following proposition characterizes 2-orbifolds that admit Besse metrics.

**Proposition 2.4.** *A 2-orbifold admits a Besse metric if and only if it is either bad or spherical, i.e. if and only if its orbifold Euler characteristic is positive.*

*Proof.* Suppose that  $\mathcal{O}$  is a Besse 2-orbifold with isolated singularities. Then the unit tangent bundle  $M = T^1\mathcal{O}$  is an orientable manifold (cf. [ALR07]). It inherits an orientation and a natural Riemannian metric (Sasaki metric) from  $\mathcal{O}$  (cf. [Be78, Ch. 1.K]) with respect to which the integral curves of the geodesic field on  $M$  are geodesics that project to the geodesics on  $\mathcal{O}$  with the same arc-length parametrization and the same period. All the geodesics on  $\mathcal{O}$  are obtained in this way. It follows from Wadsley's result (cf. Section 2.5) that these integral curves on  $M$  have a common period, say  $l$ , and thus so have the geodesics on  $\mathcal{O}$ . Since  $\mathcal{O}$  is geodesically complete every pair of points on  $\mathcal{O}$  can be connected by a (minimizing and hence orbifold) geodesic due to the Hopf-Rinow theorem [BBI01, Thm. 2.5.28]. It follows that  $\text{diam}(\mathcal{O}) \leq l$  and thus that  $\mathcal{O}$  is compact [BBI01, Thm. 2.5.28]. Suppose, in addition, that  $\mathcal{O}$  is good. Then it is finitely covered by a Besse manifold, since every good compact 2-orbifold is finitely covered by a manifold [Sc03, Thm. 2.5]. Since the fundamental group of a Besse manifold is finite [Be78, Thm. 7.37],  $\mathcal{O}$  must be spherical. For a Besse 2-orbifold whose

singular points are not isolated the same argument applies to its metric double, which is a Besse 2-orbifold with isolated singularities.  $\square$

**2.7. Orbifolds of geodesics and geodesic period spectra.** Suppose that  $\mathcal{O}$  is a Besse 2-orbifold with only isolated singularities. This is in particular the case if  $\mathcal{O}$  is orientable. Then  $M = T^1\mathcal{O}$  is a manifold and the geodesic flow on it is periodic due to theorems by Epstein [Ep72] or Wadsley [Wa75] as seen above. Hence, the flow defines a Seifert fibering  $\mathcal{F}_g$  on  $M = T^1\mathcal{O}$  whose fibers inherit a natural orientation from the flow. The quotient  $\mathcal{O}_g = M/\mathcal{F}_g$  parametrizes the closed prime geodesics on  $\mathcal{O}$  and has a natural orbifold structure (cf. Section 2.3). A singularity on  $\mathcal{O}_g$  of order  $k$  corresponds to an exceptional prime geodesic on  $\mathcal{O}$  that is  $k$ -times shorter than the regular geodesics.

A non-orientable Besse 2-orbifold  $\mathcal{O}$  is a metric quotient of an orientable Besse 2-orbifold  $\hat{\mathcal{O}}$ , which has only isolated singularities, by an isometric orientation reversing involution  $s$ . The auxiliary metric on  $T^1\hat{\mathcal{O}}$  can be chosen to be  $s$  invariant (cf. Section 2.3) so that  $s$  induces an isometry  $s : \hat{\mathcal{O}}_g \rightarrow \hat{\mathcal{O}}_g$ . The corresponding quotient spaces are  $T^1\hat{\mathcal{O}}/s = T^1\mathcal{O}$  and  $\mathcal{O}_g = \hat{\mathcal{O}}_g/s$ . If the singularities of  $\mathcal{O}$  are not isolated then  $T^1\mathcal{O}$  has orbifold singularities, which cause additional singularities on  $\mathcal{O}_g$  that do not correspond to exceptional geodesics on  $\mathcal{O}$ . A geodesic on  $\hat{\mathcal{O}}$  projects to a shorter geodesic (of half the length) on  $\mathcal{O}$  if and only if it is invariant under  $s$  but not pointwise fixed.

Orbifolds of geodesics obtained in this way from Besse 2-orbifolds have the following additional symmetry. Consider the involution  $i : T^1\mathcal{O} \rightarrow T^1\mathcal{O}$  mapping  $(x, v)$  to  $(x, -v)$ . The involution  $i$  interchanges fibers of  $\mathcal{F}_g$  representing different orientations of the same geodesic trajectory on  $\mathcal{O}$  and reverses their natural orientation. We can suppose that the auxiliary Riemannian metric on  $T^1\mathcal{O}$  is also invariant under  $i$ . Then  $i$  induces an involutive isometry of  $\mathcal{O}_g$ . In particular, it maps singular points to singular points of the same order. A point on  $\mathcal{O}_g$  is fixed by  $i$  if and only if the corresponding geodesic on  $\mathcal{O}$  hits a codimension-0 singularity of even order or the boundary of  $|\mathcal{O}|$  perpendicularly (in the sense of centrically at the corner reflector singularities of odd order, cf. Section 2.1).

*Conventions.* From now on, by a geodesic we mean a prime geodesic with a chosen orientation. By a *geodesic of order  $k$*  we mean a geodesic which is  $k$ -times shorter than the regular geodesics. *(Geodesic) period spectra* are denoted as follows. A period spectrum of  $(\bar{2}, 3)$  for  $\mathcal{O}$  means that there are three exceptional geodesics on  $\mathcal{O}$  of order 2, 2 and 3. The bar indicates that the geodesics of order 2 can be traversed with two different orientations that are interchanged by  $i$ , i.e. the corresponding points on  $\mathcal{O}_g$  project to a single point on  $\mathcal{O}_g/i$ . The other exceptional geodesic of order 3 admits only one orientation. The corresponding point on  $\mathcal{O}_g$  is fixed by  $i$ .

**2.8. Finite group actions on  $S^2$ .** We will encounter isometric actions of finite groups on Riemannian 2-orbifolds  $\mathcal{O}$  with  $|\mathcal{O}| = S^2$ . Such an action can be smoothed, i.e. there exists a smooth structure on  $|\mathcal{O}|$  with respect to which the group acts smoothly. Indeed, the orbifold admits an equivariant triangulation and the corresponding simplicial complex can be equivariantly smoothed [La16]. Then it follows from the classification of 2-orbifolds with positive Euler characteristic (cf. [Da11]) that the action can be conjugated to a linear action on  $S^2$  (cf. [Zim12]).

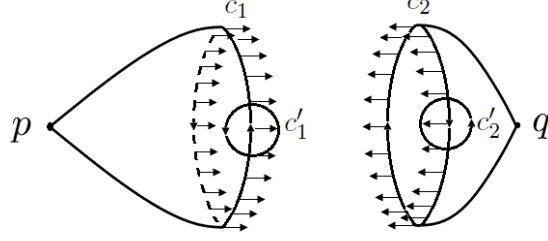


FIGURE 2. Gluing construction in Lemma 3.1. Note that the curve  $c_2$  is homotopic to the curve obtained from  $c$  by reversing the direction of the arrows from left to right.

### 3. PROOF OF THE MAIN RESULT

We first treat the cases of spindle orbifolds and the real projectiv plane. Proving our result in these cases is an essential part of proving it in all the other cases (see below).

**3.1. Spindle orbifolds.** Let  $\mathcal{O}$  be a Besse  $(p, q)$ -spindle orbifold. Recall that we do not make assumptions on  $\gcd(p, q)$ . We claim that  $\mathcal{O}_g = S^2((p+q)/\kappa, (p+q)/\kappa)$ , where  $\kappa$  is 1 or 2 depending on whether  $p+q$  is odd or even, and that the geodesic period spectrum is  $((p+q)/\kappa)$ . In other words, there exist two exceptional geodesics on  $\mathcal{O}$  with the same trajectory that are  $(p+q)/\kappa$ -times shorter than the regular geodesics. The proof is divided into steps a)-f).

a) In Section 2.7 we have seen that the unit tangent bundle  $M = T^1\mathcal{O}$  is a manifold.

**Lemma 3.1.** *The unit tangent bundle  $M = T^1\mathcal{O}$  of a  $(p, q)$ -spindle orbifold  $\mathcal{O}$  is a lens space. More precisely, we have  $M \cong L(p+q, 1)$ .*

*Proof.* To prove the lemma we cut  $M$  along the preimage of an equator of  $\mathcal{O}$  that separates the two singular points into two pieces. This preimage is a torus and the obtained pieces  $T_i$ ,  $i = 1, 2$ , admit global orbifold charts. In fact, they are quotients of full tori  $\tilde{T}_i = T^1D^2 = D^2 \times S^1$ ,  $i = 1, 2$ , by cyclic groups of order  $p$  and  $q$ , respectively, acting in both factors in a standard way. Hence the  $T_i$  are full tori themselves. In particular,  $M$  can be recovered from these full tori by specifying the gluing homeomorphism  $\psi : \partial T_1 \rightarrow \partial T_2$ . The homeomorphism type of  $T_1 \cup_\psi T_2$  is determined by the homology class of  $\psi(m_1)$  in  $\partial T_2$  for a meridian  $m_1$  of  $\partial T_1$ . We define two loops  $\tilde{c}_1, \tilde{c}'_1 : S^1 \rightarrow \partial \tilde{T}_1 = S^1 \times S^1$  by  $\tilde{c}_1(z) = (z, z)$  and  $\tilde{c}'_1(z) = (1, z)$  and  $\tilde{c}_2, \tilde{c}'_2 : S^1 \rightarrow \partial \tilde{T}_2$  analogously. We can choose meridians  $\tilde{m}_i$  of  $\tilde{T}_i$  that project to meridians  $m_i$  of  $T_i$ . We choose the orientations of  $\tilde{m}_i$  such that in homology of  $\partial \tilde{T}_i$  we have  $\tilde{m}_i \sim -\tilde{c}_i + \tilde{c}'_i$ . Let  $c'_i : S^1 \rightarrow \partial T_i$  be the projection of  $\tilde{c}'_i$  and let  $c_i : S^1 \rightarrow \partial T_i$  be curves such that  $\tilde{c}_i$  projects to  $r_i \cdot c_i$ , where  $r_1 = p$  and  $r_2 = q$ . Then, in homology we have  $m_i \sim -r_i c_i + c'_i$ . Observe that we recover  $M$  if the attaching map  $\psi$  satisfies  $\psi(c_1) \sim -c_2$  and  $\psi(c'_1) \sim c'_2$  (cf. Figure 2). Picking  $l_2 = c_2$  as a longitude in  $\partial T_2$  we have  $\psi(m_1) \sim pc_2 + c'_2 = 1 \cdot m_2 + (p+q) \cdot l_2$ . The resulting space  $T_1 \cup_\psi T_2$  is a lens space of type  $L(p+q, 1)$  as claimed (cf. Section 2.4).  $\square$

b) Recall from Section 2.7 that  $\mathcal{F}_g$  denotes the Seifert fibering on  $M = T^1\mathcal{O}$  defined by the geodesic flow. As an auxiliary tool we also need the Seifert fibering  $\mathcal{F}_t$  on  $M$  induced by the



natural projection  $M = T^1\mathcal{O} \rightarrow \mathcal{O}$ . We have  $M/\mathcal{F}_t = \mathcal{O}$  as Riemannian orbifolds. The lifts of  $\mathcal{F}_t$  and  $\mathcal{F}_g$  to  $\tilde{M}$  define Seifert fiberings of  $\tilde{M}$ . We denote them by  $\tilde{\mathcal{F}}_t$  and  $\tilde{\mathcal{F}}_g$  and the corresponding quotients by  $\tilde{M}/\tilde{\mathcal{F}}_t = \tilde{\mathcal{O}}_t$  and  $\tilde{M}/\tilde{\mathcal{F}}_g = \tilde{\mathcal{O}}_g$ . The fiberings  $\mathcal{F}_t$  and  $\mathcal{F}_g$  on  $M$  as well as their lifts on  $\tilde{M}$  are fiberwise transversal.

c) We have the following commutative diagram

$$\begin{array}{ccccc} \tilde{\mathcal{O}}_t & \xleftarrow{\quad} & \tilde{M} \cong S^3 & \xrightarrow{\quad} & \tilde{\mathcal{O}}_g \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O} \cong S^2(p, q) & \xleftarrow{\quad} & M \cong L(p + q, 1) & \xrightarrow{\quad} & \mathcal{O}_g \end{array}$$

where the outer vertical projections are coverings of Riemannian orbifolds. The upper horizontal projections induce surjections  $\pi_1(\tilde{M}) \rightarrow \pi_1^{orb}(\tilde{\mathcal{O}}_t)$  and  $\pi_1(\tilde{M}) \rightarrow \pi_1^{orb}(\tilde{\mathcal{O}}_g)$  (cf. [Sc03, Lem. 3.2]). According to the classification of simply connected compact 2-orbifolds (cf. [Sc03, Thm. 2.5]) this implies  $\tilde{\mathcal{O}}_g \cong S^2(p_g, q_g)$  and  $\tilde{\mathcal{O}}_t \cong S^2(p_t, q_t)$  as orbifolds for coprime  $p_g, q_g$  and coprime  $p_t, q_t$ . In other words, Seifert fiberings on  $S^3$  are uniquely determined by two coprime positive integers [Sei33]. Let  $\Gamma \cong \mathbb{Z}_{p+q}$  be the group of Deck transformations of the covering  $\tilde{M} \rightarrow M$ . The action of  $\Gamma$  on  $\tilde{M}$  induces actions on  $\tilde{\mathcal{O}}_t$  and  $\tilde{\mathcal{O}}_g$  which are not necessarily effective. Denote by  $\Gamma_t$  and  $\Gamma_g$  the quotients of  $\Gamma$  by the respective kernels. The groups  $\Gamma_t$  and  $\Gamma_g$  are cyclic and we have  $\tilde{\mathcal{O}}_t/\Gamma_t = \mathcal{O}$  and  $\tilde{\mathcal{O}}_g/\Gamma_g = \mathcal{O}_g$ . Since  $\Gamma$  preserves the orientation of  $\tilde{M}$  and the orientations of the fibers of  $\mathcal{F}_t$  and  $\mathcal{F}_g$ , the groups  $\Gamma_t$  and  $\Gamma_g$  preserve the orientation of  $|\tilde{\mathcal{O}}_t| \cong S^2$  and  $|\tilde{\mathcal{O}}_g| \cong S^2$ , respectively. Therefore, the action of  $\Gamma_t$  and of  $\Gamma_g$  can be conjugated to a standard action of a cyclic group on  $S^2$  (cf. Section 2.8). Moreover, since  $\Gamma$  acts isometrically on  $\tilde{\mathcal{O}}_t$  and  $\tilde{\mathcal{O}}_g$  with respect to the orbifold metrics introduced above, it fixes the singular points. Both together implies  $p = |\Gamma_t|p_t$ ,  $q = |\Gamma_t|q_t$  (up to permutation) and  $\mathcal{O}_g = S^2(|\Gamma_g|p_g, |\Gamma_g|q_g)$ .

d) If  $pq$  is odd, then the involution  $i : \mathcal{O}_g \rightarrow \mathcal{O}_g$  introduced in Section 2.7 has no fixed points. Hence, in this case it follows that  $|\Gamma_g|p_g = |\Gamma_g|q_g$  and thus  $p_g = q_g = 1$  since  $p_g$  and  $q_g$  are coprime. For even  $pq$  the same conclusion will follow from the subsequent lemma. Let  $\tilde{i} : \tilde{M} \rightarrow \tilde{M}$  be a lift of  $i : M \rightarrow M$ .

**Lemma 3.2.** *The actions of  $\tilde{i}$  and  $\Gamma$  on  $\tilde{M} = S^3$  commute.*

*Proof.* Let  $S_x^1$  be a  $\Gamma$ -invariant fiber of  $\tilde{\mathcal{F}}_t$  and let  $\gamma \in \Gamma$ . The fiber  $S_x^1$  is also fixed by  $\tilde{i}$  and both  $\gamma$  and  $\tilde{i}$  preserve the orientation of  $S_x^1$ . Since  $\Gamma$  is normalized by  $\tilde{i}$ , it follows that  $\gamma$  and  $\tilde{i}$  generate a finite group that preserves the the orientation of  $S_x^1$ . Such a group must be abelian and thus  $\gamma$  and  $\tilde{i}$  commute on  $S_x^1$ . Since their commutator  $\gamma\tilde{i}\gamma^{-1}\tilde{i}^{-1} : \tilde{M} \rightarrow \tilde{M}$  is a lift of the identity of  $M$ ,  $\gamma$  and  $\tilde{i}$  commute everywhere. The claim follows.  $\square$

Indeed, now we can show

**Lemma 3.3.** *The involution  $i$  does not fix singular points on  $\mathcal{O}_g$ .*

*Proof.* We only need to consider the case that  $pq$  is even. Suppose a singular point on  $\mathcal{O}_g$  is fixed by  $i$ . We have seen above that the corresponding fiber  $S_c^1$  of  $\tilde{\mathcal{F}}_g$  and its orientation

are  $\Gamma$ -invariant. By assumption, the map  $\tilde{i}$  also leaves the fiber  $S_c^1$  invariant but reverses its orientation. Therefore,  $\Gamma$  and  $\tilde{i}$  generate a dihedral group. Since  $\Gamma$  and  $\tilde{i}$  commute by Lemma 3.2 it follows that  $2 \leq p + q = |\Gamma| \leq 2$ . This contradicts the existence of a singular point on  $\mathcal{O}$  of even order and thus the claim follows.  $\square$

Consequently, in any case we have  $\mathcal{O}_g = S^2(|\Gamma_g|, |\Gamma_g|)$  and so it remains to determine the order of  $\Gamma_g$ .

e) An example of a Seifert fibering on  $S^3$  is the Hopf fibration  $\mathcal{H}$  defined by the free  $S^1$ -action  $\varphi^+$  (or  $\varphi^-$ )

$$\begin{aligned} \varphi^\pm : S^1 \times S^3 &\longrightarrow S^3 \\ (e^{it}, (z_1, z_2)) &\longmapsto (e^{it}z_1, e^{\pm it}z_2). \end{aligned}$$

Since the actions of  $\varphi^+$  and  $\varphi^-$  commute, they induce almost-free actions  $\varphi^\pm : S^1 \times L(r, 1) \longrightarrow L(r, 1)$  and Seifert fiberings  $\mathcal{F}^\pm$  on  $L(r, 1)$  where  $L(r, 1)$  is the quotient of  $S^3$  by the action of  $\varphi^+$  restricted to the  $r$ -th roots of unity in  $S^1$ . The following lemma shows that these are essentially the only actions that can occur in our situation.

**Lemma 3.4.** *Let  $\varphi : S^1 \times L(r, 1) \longrightarrow L(r, 1)$ ,  $r > 1$ , be a smooth, almost-free action with quotient space homeomorphic to  $S^2$ . If there are no exceptional fibers, then  $\varphi$  is smoothly conjugated to  $\varphi^+$ . If there are precisely two exceptional fibers of the same order  $k$ , then  $\varphi$  is smoothly conjugated to  $\varphi^-$ . In the latter case we have  $r = \kappa k$ , where  $\kappa$  is 1 or 2 depending on whether  $r$  is odd or even. In particular,  $r$  is divisible by 4 if  $k$  is even.*

*Proof.* Let  $\mathcal{F}$  be the Seifert fibering defined by  $\varphi$ . We consider both cases at the same time by setting  $k = 1$  if there are no exceptional fibers. It is sufficient to show that the Seifert invariants of  $\mathcal{F}$  coincide up to orientation with either those of  $\mathcal{F}^+$  or  $\mathcal{F}^-$  (cf. Section 2.3).

Since  $L(r, 1)/\mathcal{F}$  is a 2-sphere by assumption, the Seifert fiber space  $(L(r, 1), \mathcal{F})$  can be obtained as  $T_1 \cup_\psi T_2$  where  $T_1$  and  $T_2$  are fibered solid tori with an exceptional fiber of order  $k$  and where  $\psi : \partial T_1 \rightarrow \partial T_2$  is an orientation-reversing and fiber-preserving homeomorphism (cf. [Br93, Thm. 1.4.5.]). Choose meridians  $m_1$  and  $m_2$  on  $\partial T_1$  and  $\partial T_2$  and a longitude  $l_2$  on  $\partial T_2$ . In homology we have  $\psi(m_1) \sim s'm_2 + r'l_2$  for some integers  $s', r'$ . Since  $T_1 \cup_\psi T_2$  is a  $L(r, 1)$  lens space with  $r > 1$  by assumption, we have  $r' = \varepsilon_1 r$  and  $s' = \varepsilon_2 + tr \neq 0$  for some integer  $t$  and some  $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$  (cf. Section 2.4). Replacing  $l_2$  by  $tm_2 + \varepsilon_1 l_2$  and  $m_2$  by  $\varepsilon_2 m_2$  we can assume that  $\psi(m_1) \sim m_2 + rl_2$ . In particular,  $l_1 = \psi^{-1}(l_2)$  is a longitude of  $\partial T_2$ . Let  $f_2$  be a (regular) fiber on  $\partial T_2$ . Perhaps after reversing the orientation of  $f_2$  we have  $f_2 \sim b_2 m_2 + kl_2$  for some integer  $b_2$ . The preimage  $f_1 = \psi^{-1}(f_2)$  is a regular fiber on  $\partial T_1$  and we have  $f_1 \sim b_1 m_1 + \varepsilon k l_1$  for some integer  $b_1$  and  $\varepsilon \in \{\pm 1\}$ . Since  $f_1$  and  $f_2$  are without self-intersections, we have  $\gcd(b_1, k) = \gcd(b_2, k) = 1$ . Now the conditions  $\psi(m_1) \sim m_2 + rl_2$ ,  $\psi(l_1) \sim l_2$  and  $\psi(f_1) = f_2$  imply that  $b_1 = b_2$  and  $rb_1 = k(1 - \varepsilon)$ . Hence, because of  $\gcd(b_1, k) = 1$ , we are in one of the following three mutually exclusive cases

- (i)  $\varepsilon = 1$ ,  $k = 1$ ,  $b_1 = b_2 = 0$ .
- (ii)  $\varepsilon = -1$ ,  $2k = r$  even, and  $b_1 = b_2 = 1$ .
- (iii)  $\varepsilon = -1$ ,  $k = r$  odd, and  $b_1 = b_2 = 2$ .

Since these data completely determine the fiber-homomorphism type of  $T_1 \cup_\psi T_2$  and since the same argument applies to  $(L(r, 1), \mathcal{F}^\pm)$  the claim follows.  $\square$

Alternatively, the lemma can be shown as follows. A Seifert fiber space of the form  $T_1 \cup_\psi T_2$  as in the lemma can be written as  $M(0; (k, \beta_1), (\pm k, \beta_2))$  (type  $o_1$ , genus 0) with intergers  $\beta_1, \beta_2$ . Such a space is a lens space  $L(r, s)$  with

$$\begin{aligned} r &= k(\beta_2 \pm \beta_1) \\ s &= k\beta'_2 + \beta_1\alpha'_2 \end{aligned}$$

where  $\pm k\beta'_2 - \beta_2\alpha'_2 = 1$  by [JN83, Thm. 4.4]. A case differentiation shows that for  $s = 1$  we are in one of the following three cases (cf. appendix, Section 4.3)

- (i)  $(L(r, 1), \mathcal{F}) = M(0; (1, r)) = (L(r, 1), \mathcal{F}^+)$  with  $k = 1$
- (ii)  $(L(r, 1), \mathcal{F}) = M(0; (k, 1), (k, 1)) = (L(r, 1), \mathcal{F}^-)$  with  $r = 2k$  even
- (iii)  $(L(r, 1), \mathcal{F}) = M(0; (k, \frac{1+k}{2}), (k, \frac{1-k}{2})) = (L(r, 1), \mathcal{F}^-)$  with  $r = k$  odd

and these cases correspond to the three cases described in the lemma.

f) In case of a sphere, i.e.  $p + q = 2$ , we have  $|\Gamma_g| \in \{1, 2\}$  and thus  $|\Gamma_g| = 1$  and  $\mathcal{O}_g = S^2$  by the lemma above. Suppose that  $p + q > 2$ . According to the last paragraph  $(M, \mathcal{F}_g)$  is either homeomorphic to  $(L(p + q), \mathcal{F}^+)$  or to  $(L(p + q), \mathcal{F}^-)$ . Hence, we can assume that  $\tilde{\mathcal{F}}_g$  is the Hopf fibration  $\mathcal{H}$  on  $S^3$  defined by  $\varphi^+$  and that  $\Gamma \cong \mathbb{Z}_{p+q}$  acts linearly via  $\varphi^+$  or  $\varphi^-$ . Let  $S_g^1$  be a  $\Gamma$ -invariant fiber of  $\tilde{\mathcal{F}}_g$ . Recall that the fibers of  $\tilde{\mathcal{F}}_g$  come along with a natural orientation and that the map  $\tilde{i} : S_g^1 \rightarrow \tilde{i}(S_g^1)$  reverses the orientation. Let  $\gamma$  be a generator of  $\Gamma$  that rotates  $S_g^1$  about a minimal angle. Since  $\Gamma$  and  $\tilde{i}$  commute by Lemma 3.2, it follows that  $\gamma$  rotates  $S_g^1$  and  $\tilde{i}(S_g^1)$  in different directions with respect to their natural orientations. Now the fact that the  $\mathbb{Z}_{p+q}$  action on  $(S^3, \mathcal{H})$  via  $\varphi^+$  rotates all fibers in the same direction implies that  $\mathcal{F}_g = \mathcal{F}^-$  in view of Lemma 3.4. In other words, there are two geodesics on  $\mathcal{O}$  that are  $(p + q)/\kappa$ -times shorter than all the other geodesics and they are interchanged by  $i$  due to Lemma 3.3.

Observe that in the case  $p = q$  we have  $\mathcal{O}_g \cong S^2/C_p$  and  $\mathcal{O}_g/i \cong S^2/\langle C_p, -1 \rangle$ , where  $C_p < \text{SO}(3)$  is a cyclic group of order  $p$ .

**3.2. The real projective plane.** In this section we apply the above analysis to the real projective plane  $\mathcal{O} = \mathbb{RP}^2$  endowed with a Besse metric. The meaning of the notations  $\mathcal{F}_g, \mathcal{F}_t, \tilde{\mathcal{F}}_g$  and  $\tilde{\mathcal{F}}_t$  is analogous to those in Section 3.1 (cf. step b)). The unit tangent bundle  $M$  of  $\mathbb{RP}^2$  is homeomorphic to the lens space  $L(4, 1)$  [Ko02]. We claim that  $\mathcal{O}_g = S^2$  and thus that all geodesics on  $\mathbb{RP}^2$  have the same period. This was already shown in [Pr09] in a different way. The claim will follow as in paragraph f) of Section 3.1, if we can show that a lift  $\tilde{i} : \tilde{M} \rightarrow \tilde{M}$  of the natural involution  $i : T^1\mathbb{RP}^2 \rightarrow T^1\mathbb{RP}^2$  to the universal cover  $\tilde{M} \cong S^2$  of  $M$  does not commute with  $\Gamma = \text{Deck}(\tilde{M} \rightarrow M) \cong \mathbb{Z}_4$ . In fact, in this case  $\Gamma$  leaves the fibers of  $\tilde{\mathcal{F}}_g$  invariant and thus the induced action on  $\tilde{M}/\tilde{\mathcal{F}}_g \cong S^2$  is trivial. The involution  $i : M \rightarrow M$  lifts to the natural involution  $i : T^1S^2 \rightarrow T^1S^2$ . We take  $\tilde{i}$  to be a lift of  $i : T^1S^2 \rightarrow T^1S^2$ .

**Lemma 3.5.** *The actions of  $\tilde{i}$  and  $\Gamma$  on  $\tilde{M} = S^3$  do not commute.*

*Proof.* Since the preimages of the fibers of  $T^1S^2 \rightarrow S^2$  under the twofold covering  $\tilde{M} = S^3 \rightarrow T^1S^2$ , i.e. the fibers of  $\tilde{\mathcal{F}}_t \cong \mathcal{H}$ , are connected, the lift  $\tilde{i} : \tilde{M} \rightarrow \tilde{M}$  leaves them invariant and thus has order 4. The fibers of  $\tilde{\mathcal{F}}_t$  inherit orientations from the fibers of  $T^1S^2 \rightarrow S^2$ , which

are preserved by  $\tilde{i}$  and reversed by  $\Gamma$ . Let  $S_{t,1}^1, S_{t,2}^1$  be a pair of fibers of  $\tilde{\mathcal{F}}_t$  that is invariant under  $\Gamma = \langle \gamma \rangle \cong \mathbb{Z}_4$ . Since both  $S_{t,1}^1$  and  $S_{t,2}^1$  are  $\tilde{i}$ -invariant and since the map  $\gamma : S_{t,1}^1 \rightarrow S_{t,2}^1$  reverses the orientation while  $\tilde{i}$  preserves it, we see that  $\Gamma$  and  $\tilde{i}$  cannot commute.  $\square$

In the appendix, Section 4.2, we show how this result can be used to deduce rigidity on the real projective plane. In particular, this implies that every Besse orbifold covered by the real projective plane has constant curvature.

**3.3. Non-orientable half-spindle orbifolds.** Let us suppose that  $\mathcal{O}$  is a Besse 2-orbifold of type  $D^2(;p,q)$ . Let  $\hat{\mathcal{O}}$  be its orientable double cover of type  $S^2(p,q)$  and let  $s$  be the Deck transformation of the covering  $\hat{\mathcal{O}} \rightarrow \mathcal{O}$ . From Section 3.1 we know that  $\hat{\mathcal{O}}_g \cong S^2((p+q)/\kappa, (p+q)/\kappa)$  where  $\kappa$  is 1 or 2 depending on whether  $p+q$  is odd or even. Since the action of  $s$  on  $\hat{\mathcal{O}}_g$  preserves the orientation and can be linearized, it is either trivial or fixes precisely two points. We claim that the latter is always the case. Geodesics hitting singular points on  $\hat{\mathcal{O}}$  are regular due to Lemma 3.3. We can choose a (regular) geodesic  $c$  on  $\hat{\mathcal{O}}$  starting at the singularity of order  $p$  whose initial direction  $v$  is almost fixed by  $s$  but not quite. By continuity  $c$  stays close to a (regular) geodesic in  $\text{Fix}(s) \subset \hat{\mathcal{O}}$ . Differing cases depending on the parity of  $p$  and  $q$  it follows that  $c$  cannot be invariant under  $s$  (in a neighborhood of the  $p$ -singularity). Hence,  $s$  fixes precisely two points on  $\hat{\mathcal{O}}_g$ . Among them are the geodesics contained in  $\text{Fix}(s) \subset \hat{\mathcal{O}}$ . If  $p+q$  is even, there are precisely two such geodesics and they are pointwise fixed by  $s$ . In this case the period spectrum is  $((p+q)/2)$  and we have  $\mathcal{O}_g = S^2(2, 2, (p+q)/2)$ . If  $p+q$  is odd there is only one geodesic contained in  $\text{Fix}(s)$ , which is again pointwise fixed by  $s$ . The other  $s$ -invariant geodesic must be regular, cannot be pointwise fixed and thus projects to a geodesic of half the period. Hence, in this case the period spectrum is  $(2, (p+q))$  and we have  $\mathcal{O}_g = S^2(2, 2, (p+q))$ .

**3.4. Orientable spherical orbifolds.** The orientable spherical orbifolds correspond to the finite subgroups of  $\text{SO}(3)$ . These are cyclic groups  $C_n$  of order  $n$ , dihedral groups  $D_n$  of order  $2n$  and tetrahedral, octahedral and icosahedral groups  $T$ ,  $O$  and  $I$  of order 12, 24 and 60, respectively. The case of  $C_n$  has been treated in Section 3.1. Let us suppose that  $\mathcal{O}$  is a quotient of  $S^2$  by one of the other groups, call it  $G$ , endowed with a Besse metric. In other words we have a  $G$ -invariant Besse metric on  $S^2$  and we denote this Besse manifold by  $\tilde{\mathcal{O}}$ . From Section 3.1 we know that  $\tilde{\mathcal{O}}_g \cong S^2$  as orbifolds. Since  $G$  is generated by rotations the preceding section moreover shows that  $G$  acts effectively on  $\tilde{\mathcal{O}}_g$  and preserves the orientation. This action can be linearized (cf. Section 2.8). Since the faithful representation of  $G$  in  $\text{SO}(3)$  is unique, it follows that  $\mathcal{O}_g = \tilde{\mathcal{O}}_g/G \cong \mathcal{O}$  as orbifolds. By Lemma 3.2 the actions of  $G$  and  $i$  on  $\tilde{\mathcal{O}}_g$  commute. Since the representation of  $G$  in  $\text{SO}(3)$  is absolutely irreducible, Schur's lemma implies that  $\mathcal{O}_g/i \cong S^2/G^*$  with  $G^* = \langle G, -1 \rangle$ . Recall from the end of Section 3.1 that the same expressions hold in the case of  $C_n$ . This proves the data listed in Table 1 under number 2.-6.. For convenience and later usage we consider the cases of  $D_n$  and  $T$  in more detail.

3., 4. If  $G = D_n$ ,  $n \geq 2$ , then  $\mathcal{O}, \mathcal{O}_g \cong S^2(2, 2, n)$  have a double cover  $\hat{\mathcal{O}}, \hat{\mathcal{O}}_g \cong S^2(n, n)$ . For even  $n$  there exist two geodesics on  $\hat{\mathcal{O}}$  connecting the two singular points on  $\hat{\mathcal{O}}$  of order  $n$  and passing a branch point of the covering  $\hat{\mathcal{O}} \rightarrow \mathcal{O}$ . In fact, we can choose minimizing

segments between one of these singularities and the two branch points. Since the covering is twofold these segments extend to trajectories of the desired geodesics. By construction, these geodesics project to exceptional geodesics on  $\mathcal{O}$  of half the (regular) period that are fixed by  $i$ . Hence, we have  $\mathcal{O}_g/i \cong D^2(; 2, 2, n)$  and thus the period spectrum is  $(2, 2, n)$ . For odd  $n$  geodesics on  $\hat{\mathcal{O}}$  passing a branch point of the covering  $\hat{\mathcal{O}} \rightarrow \mathcal{O}$  project to geodesics on  $\mathcal{O}$  of the same period and thus in this case we have  $\mathcal{O}_g/i \cong D^2(2; n)$  and thus the period spectrum is  $(\bar{2}, n)$ . In both cases a geodesic of order  $n$  on  $\hat{\mathcal{O}}$  hits one of the branch points of  $\hat{\mathcal{O}} \rightarrow \mathcal{O}$ . (In fact, its projection to  $\mathcal{O}$  oscillates between the two new singularities of order 2, but we will not need this statement).

5. For  $G = T$  a geodesic  $c$  of order 3 on  $\mathcal{O}$  lifts to a geodesic on  $\tilde{\mathcal{O}} \cong S^2$  that is invariant under a subgroup of  $T$  of order 3. If  $c$  hit a singularity of order 2 on  $\mathcal{O}$ , its lift would be invariant under a subgroup of  $T$  of order  $\geq 6$  contradicting the fact that  $G$  acts effectively on  $\tilde{\mathcal{O}}_g \cong S^2$  with quotient  $S^2(2, 3, 3)$ . Hence, we have  $\mathcal{O}_g/i \cong D^2(3; 2)$  and thus the period spectrum is  $(2, \bar{3})$ .

**3.5. Non-orientable spherical orbifolds.** Suppose that  $\mathcal{O}$  is a quotient of  $S^2$  by a finite subgroup  $G$  of  $O(3)$  that does not preserve the orientation. Suppose further that  $\mathcal{O}$  is endowed with a Besse metric. In other words we have a  $G$ -invariant Besse metric on  $S^2$  and we denote this Besse manifold by  $\tilde{\mathcal{O}}$ . Let  $G^+$  be the orientation preserving subgroup of  $G$  and let  $\hat{\mathcal{O}} = \tilde{\mathcal{O}}/G^+$ . We have seen that  $\tilde{\mathcal{O}}_g \cong S^2$  and that  $\hat{\mathcal{O}}_g \cong \tilde{\mathcal{O}}/G^+$  as orbifolds. For  $-1 \in G$  we have seen in Section 3.2 that  $\mathcal{O}_g \cong \mathcal{O}/G^+$  as orbifolds and that  $\mathcal{O}$  has constant curvature. For  $-1 \notin G$  we claim that  $G$  acts effectively on  $\tilde{\mathcal{O}}_g$ . Indeed, in this case the  $\text{ord}(g)/2$ -th power of an element  $g \in G$  with  $\det(g) = -1$  is a reflection and this reflection does not leave invariant the geodesics that hit its fixed point set perpendicularly. Hence, in this case we have  $\mathcal{O}_g \cong S^2/G^\times$  for a finite subgroup  $G^\times$  of  $SO(3)$  that is abstractly isomorphic to  $G$  and contains  $G^+$  as an index two subgroup. It follows from the classification of finite subgroups of  $SO(3)$  that  $G^\times = \{\det(g)g | g \in G\}$ . Moreover, from the preceding section we see that  $\mathcal{O}_g/i \cong S^2/G^*$  with  $G^* = \langle G, G^+, -1 \rangle = \langle G, -1 \rangle$ .

Let  $s$  be the Deck transformation of the covering  $\hat{\mathcal{O}} \rightarrow \mathcal{O}$ . The induced action of  $s$  on  $\hat{\mathcal{O}}_g$  preserves the orientation. Since this action is linearizable (cf. Section 2.8) it is either trivial or fixes precisely two points. As seen above, it is trivial if and only if  $-1 \in G$ . In order to determine the period spectrum of  $\mathcal{O}$ , we have to identify the  $s$ -invariant geodesics on  $\mathcal{O}$  and decide whether they are pointwise fixed or not. We go through the cases with  $-1 \notin G$  listed under number 8.-12. in Table 1. The case 7. with  $G^+ = C_n$  has been treated in Section 3.3. The cases with  $-1 \in G$ , listed under number 13.-19. in Table 1, in which a Besse metric has constant curvature, can be treated analogously.

8. In the case of  $\mathcal{O} \cong \mathbb{RP}^2(2n)$  we have  $\hat{\mathcal{O}} \cong S^2(2n, 2n) \cong \hat{\mathcal{O}}_g$ ,  $\mathcal{O}_g \cong S^2(4n, 4n)$  and  $\mathcal{O}_g/i \cong D^2(4n; )$ . Hence, the period spectrum is  $(\overline{4n})$ .

9. In the case of  $\mathcal{O} \cong D^2(2n+1; )$  we have  $\hat{\mathcal{O}} \cong S^2(2n+1, 2n+1) \cong \hat{\mathcal{O}}_g$ ,  $\mathcal{O}_g \cong S^2(4n+2, 4n+2)$  and  $\mathcal{O}_g \cong D^2(4n+2; )$ . The exceptional geodesics on  $\hat{\mathcal{O}}$  are contained in the fixed point set of  $s$  and thus the period spectrum is  $(\overline{2n+1})$ .

10. In the case of  $\mathcal{O} \cong D^2(2; 2n)$  we have  $\hat{\mathcal{O}} \cong S^2(2, 2, 2n) \cong \hat{\mathcal{O}}_g$ ,  $\mathcal{O}_g \cong S^2(2, 2, 4n)$  and  $\mathcal{O}_g/i \cong D^2(2; 2, 2, 4n)$ . Invariant under  $s$  are an exceptional geodesic of order  $2n$  and a regular geodesic. In Section 3.4 we have seen that this exceptional geodesic of order  $2n$  hits a singularity of order 2 that is not fixed by  $s$ . Therefore, the invariant exceptional geodesic is not pointwise fixed, only the regular geodesic in  $\text{Fix}(s) \subset \hat{\mathcal{O}}$  is so. We conclude that the period spectrum is  $(2, 4n)$ .

11. In the case of  $\mathcal{O} \cong D^2(2; 2, 2, 2n+1)$  we have  $\hat{\mathcal{O}} \cong S^2(2, 2, 2n+1) \cong \hat{\mathcal{O}}_g$ ,  $\mathcal{O}_g \cong S^2(2, 2, 4n+2)$  and  $\mathcal{O}_g/i \cong D^2(2; 4n+2)$ . Invariant under  $s$  are the two geodesics in  $\text{Fix}(s) \subset \hat{\mathcal{O}}$  which are pointwise fixed. One of them is exceptional of order  $2n+1$  and the other is regular. We conclude that the period spectrum is  $(2, 2n+1)$ .

12. In the case of  $\mathcal{O} \cong D^2(2; 3, 3)$  we have  $\hat{\mathcal{O}} \cong S^2(2, 3, 3) \cong \hat{\mathcal{O}}_g$  and  $\mathcal{O}_g \cong S^2(2, 3, 4)$ . The exceptional geodesic of order 2 on  $\hat{\mathcal{O}}$  is invariant under  $s$ . We claim that it is not pointwise fixed. To see this we consider a threefold orbifold covering  $S^2(2, 2, 2) \cong \mathcal{O}' \rightarrow \hat{\mathcal{O}} \cong S^2(2, 3, 3)$ . In Section 3.4 we have seen that the three exceptional geodesics on  $\mathcal{O}'$  are defined by minimizing segments connecting the three singularities of order 2. Due to the minimizing property their trajectories have a trivial intersection. Hence, they do not pass through the branch points of the covering  $\mathcal{O}' \rightarrow \hat{\mathcal{O}}$ . In other words, the exceptional geodesic of order 2 on  $\hat{\mathcal{O}}$  does not hit a singularity of order 3 and is thus not contained in  $\text{Fix}(s)$ . The other invariant geodesic is the regular geodesic in  $\text{Fix}(s) \subset \hat{\mathcal{O}}$  which is pointwise fixed. We conclude that the period spectrum is  $(3, 4)$ .

#### 4. APPENDIX

**4.1. Conjugacy of induced contact structures.** Let  $\mathcal{O}$  be a Besse 2-orbifold with only isolated singularities. As in the manifold case (cf. [ABHS16], Appendix B and [Ge08, Thm. 1.5.2]) the unit tangent bundle  $M = T^1\mathcal{O}$  carries a natural contact 1-form  $\alpha$  whose Reeb flow is the geodesic flow on  $M$ . The form  $\alpha$  is the restriction of the pullback of the canonical Liouville form on  $T^*\mathcal{O} - \mathcal{O}$  via the isomorphism  $T\mathcal{O} - \mathcal{O} \rightarrow T^*\mathcal{O} - \mathcal{O}$  induced by the metric (here we regard  $\mathcal{O} \subset T^{(*)}\mathcal{O}$  as the zero-section). The aim of this section is to prove the following result.

**Theorem 4.1.** *Let  $g_0$  and  $g$  Besse metrics on a 2-orbifold  $\mathcal{O}$  with only isolated singularities and let  $\alpha_0$  and  $\alpha$  be the corresponding contact forms on  $T^1(\mathcal{O}, g_0)$  and  $T^1(\mathcal{O}, g)$ , respectively. Then there exists a diffeomorphism*

$$\varphi : T^1(\mathcal{O}, g_0) \rightarrow T^1(\mathcal{O}, g)$$

*such that  $\varphi^*\alpha = \alpha_0$ .*

*Proof.* For the 2-sphere this result is proven in [ABHS16], Appendix B. We explain how the same methods can be used to prove the general case.

In case of the real projective plane and of spindle orbifolds we have seen in Lemma 3.4 that the Seifert fiber type of the induced  $S^1$ -action on  $T^1\mathcal{O}$  is unique up to orientation and thus that all such actions are conjugated by a diffeomorphism (cf. Section 2.3). The same statement is true for the remaining orbifolds in question, i.e. the platonic orbifolds  $S^2(2, 2, n)$ ,  $S^2(2, 3, 3)$ ,

$S^2(2, 3, 4)$  and  $S^2(2, 3, 5)$ , since their unit tangent bundles admit up to orientation only one Seifert fibering with appropriate orbit surface [Or72, Thm. 2, p. 111].

Hence, in each case there exists a diffeomorphism  $\psi : T^1(\mathcal{O}, g_0) \rightarrow T^1(\mathcal{O}, g)$  that conjugates the  $S^1$  geodesic flow actions. We claim that such a diffeomorphism preserves the orientations defined by  $\alpha_0 \wedge d\alpha_0$  and  $\alpha \wedge d\alpha$ . By lifting everything to the universal cover  $S^3$  it is sufficient to show that if  $\alpha_0$  and  $\alpha$  are contact forms on  $S^3$  with identical Reeb fields such that the Reeb flow is defined by an  $S^1$ -action on  $S^3$ , then  $\alpha_0$  and  $\alpha$  induce the same orientation. First note that in this case all Reeb orbits have a common period, i.e. the  $S^1$ -action defines a principle  $S^1$ -bundle with base  $B = S^3/S^1 \cong S^2$  (cf. [FLS16]). By (the easy part of) a theorem by Boothby and Wang ([BW58], cf. [Ge08, Thm. 7.2.5],  $\alpha_0$  and  $\alpha$  are connection 1-forms of this principle bundle, whose curvature forms  $\omega_0$  and  $\omega$  are area forms on  $B$  satisfying  $p^*\omega_0 = d\alpha_0$  and  $p^*\omega = d\alpha$ , where  $p : S^3 \rightarrow B$  is the natural projection. Moreover,  $-\lceil \omega_0/2\pi \rceil = -\lceil \omega/2\pi \rceil$ ,  $i = 0, 1$ , is the Euler class of the principle  $S^1$ -bundle. In particular,  $\omega_0$  and  $\omega$  induce the same orientation on  $B$  and hence  $\alpha_0$  and  $\alpha$  induce the same orientation on  $S^3$ . Therefore, a diffeomorphism  $\psi : T^1(\mathcal{O}, g_0) \rightarrow T^1(\mathcal{O}, g)$  as above preserves the orientation as claimed.

Now, as in [ABHS16] one can show that  $\alpha_t = t\psi^*\alpha + (1-t)\alpha_0$  is a contact form for every  $t \in [0, 1]$  and apply Moser's argument to find a one-parameter family of diffeomorphisms  $\phi_t : T^1(\mathcal{O}, g_0) \rightarrow T^1(\mathcal{O}, g_0)$ ,  $t \in [0, 1]$ , such that  $\phi_t^*\alpha_t = \alpha_0$  for every  $t \in [0, 1]$ . In particular,

$$\alpha_0 = \phi_1^*\alpha_1 = \phi_1^*\psi^*\alpha$$

and  $\varphi = \psi \circ \phi_1$  is the desired diffeomorphism.  $\square$

**4.2. Rigidity on the real projective plane.** For a Riemannian metric  $g$  on  $\mathcal{O} = \mathbb{RP}^2$  we denote its corresponding area measure by  $\nu_g$ , its total area by  $A_g$  and the length of a shortest noncontractible loop by  $a_g$ . The following inequality due to Pu holds [Pu52]

$$A_g \geq \frac{2}{\pi} a_g^2.$$

Now suppose that  $g$  is Besse. Based on the proof of the above inequality we show that  $g$  has constant curvature. This proof was explained to us by A. Abbondandolo.

Let  $g_0$  be the standard Riemannian metric on  $\mathbb{RP}^2$  of constant curvature 1. The group  $G = \text{SO}(3)$  acts on  $\mathbb{RP}^2$  in its standard way. By the uniformization theorem there is some positive smooth function  $\varphi$  on  $\mathbb{RP}^2$  such that  $g = \varphi \cdot g_0$ . We endow  $G$  with its Haar measure  $\mu$  and define

$$\bar{\varphi} = \left( \int_G (g^*\varphi)^{1/2} d\mu \right)^2$$

and  $\bar{g} = \bar{\varphi} \cdot g_0$ . By construction  $\bar{g}$  is a  $G$ -invariant Riemannian metric on  $M = \mathbb{RP}^2$  and hence has constant curvature. We claim that  $A_{\bar{g}} \leq A_g$  and  $a_{\bar{g}} \geq a_g$ . Indeed, we have

$$\begin{aligned} A_{\bar{g}} &= \int_M \bar{\varphi} d\nu_{g_0} = \int_M \left( \int_G (h^*\varphi)^{1/2} d\mu \right)^2 d\nu_{g_0} \\ &\leq \int_M \left( \int_G h^*\varphi d\mu \right) d\nu_{g_0} = \int_G \left( \int_M h^*\varphi d\nu_{g_0} \right) d\mu \\ &= \int_G A_g d\mu = A_g \end{aligned}$$

where we have applied Cauchy-Schwarz inequality. Moreover, with a shortest noncontractible loop (and hence geodesic)  $\gamma$  on  $\mathbb{RP}^2$  with respect to  $\bar{g}$  we have

$$\begin{aligned} a_{\bar{g}} &= \int_0^1 \bar{\varphi}(\gamma(s))^{1/2} \|\dot{\gamma}(s)\|_{g_0} ds = \int_0^1 \left( \int_G ((h^*\varphi)(\gamma(s)))^{1/2} \|\dot{\gamma}(s)\|_{g_0} d\mu \right) ds \\ &= \int_G \left( \int_0^1 ((h^*\varphi)(\gamma(s)))^{1/2} \|\dot{\gamma}(s)\|_{g_0} ds \right) d\mu \\ &\geq \int_G a_g d\mu = a_g. \end{aligned}$$

For the standard metric  $g_0$  we have equality in Pu's inequality. Hence, the same is true for the Besse metric  $g$  by Theorem 4.1 and fiberwise integration. (Alternatively, this can be seen as follows: The two-fold covering  $(S^2, \hat{g})$  of  $(\mathbb{RP}^2, g)$  has area  $2A_g$  and the minimal geodesic period is  $2a_g$  due to the fact that  $\mathcal{O}_g \cong S^2$ . Now, a theorem of Weinstein says that for a Besse metric  $\hat{g}$  on  $S^2$  we have  $\text{area}(S^2, \hat{g}) = \frac{l^2}{\pi}$ , where  $l$  is the minimal geodesic period [We74], cf. [Be78, Prop. 2.24].) It follows that  $A_{\bar{g}} = A_g$ , i.e. we have equality in the Cauchy-Schwarz inequality implying that  $\varphi$  is constant. Hence,  $g$  is proportional to  $g_0$  and has constant curvature.

**4.3. Computations with Seifert invariants.** An orientable Seifert fibered space with orientable orbit surface (type  $o_1$ ) of genus  $g$  is completely characterized by a set of invariants  $M(g; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$  where  $\alpha_i, \beta_i \in \mathbb{Z}$  and every such set defines an orientable Seifert fibered space of type  $o_1$  [JN83, Thm. 1.5]. Replacing  $(\alpha_i, \beta_i), (\alpha_j, \beta_j)$ ,  $i \neq j$ , by  $(\alpha_i, \beta_i + \alpha_j), (\alpha_j, \beta_j - \alpha_j)$ ,  $(\alpha_i, \beta_i)$  by  $(-\alpha_i, -\beta_i)$  or omitting any pair  $(\alpha_i, \beta_i) = (1, 0)$  does not change the Seifert fiber type [JN83, Thm. 1.5]. Replacing every  $\beta_i$  by  $-\beta_i$  does not change the Seifert fiber type up to orientation [JN83, Cor. 1.7].

Suppose we have a Seifert fibered space of the form  $M(0; (\alpha_1, \beta_1), (\alpha_2, \beta_2))$  with integers  $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha'_2, \beta'_2$  satisfying

$$\begin{aligned} (1) \quad & r = \alpha_1 \beta_2 + \beta_1 \alpha_2 > 1 \\ & 1 = \alpha_1 \beta'_2 + \beta_1 \alpha'_2, \\ & 1 = \alpha_2 \beta'_2 - \beta_2 \alpha'_2, \end{aligned}$$

and  $|\alpha_1| = |\alpha_2| = k \geq 1$  as in Lemma 3.4. If  $\alpha_1$  and  $\alpha_2$  have the same sign, this implies  $\alpha'_2(\beta_1 + \beta_2) = 0$  and  $r = k|\beta_1 + \beta_2|$ . Hence,  $k = 1$ ,  $r = |\beta_1 + \beta_2|$  and the Seifert fiber space is  $M(0; (1, r))$ . If, on the other hand,  $\alpha_1$  and  $\alpha_2$  have opposite sign, the relations above imply

$$r = k|\beta_1 - \beta_2|, \quad \alpha'_2(\beta_1 - \beta_2) = 2 \text{ and } 2k\beta'_2 + (\beta_1 + \beta_2)\alpha'_2 = 0.$$

If, in this case, we have  $|\beta_1 - \beta_2| = 2$  and  $|\alpha'_2| = 1$ , it follows that  $r = 2k$  and the Seifert fiber space is  $M(0; (k, 1), (k, 1))$ . If, otherwise,  $|\beta_1 - \beta_2| = 1$  and  $|\alpha'_2| = 2$ , it follows that  $r = k$  and the Seifert fiber space is  $M(0; (k, \frac{1+k}{2}), (k, \frac{1-k}{2}))$ . Hence, we are in one of the following three cases as already shown in Lemma 3.4

- (i)  $(L(r, 1), \mathcal{F}) = M(0; (1, r)) = (L(r, 1), \mathcal{F}^+)$  with  $k = 1$
- (ii)  $(L(r, 1), \mathcal{F}) = M(0; (k, 1), (k, 1)) = (L(r, 1), \mathcal{F}^-)$  with  $r = 2k$  even
- (iii)  $(L(r, 1), \mathcal{F}) = M(0; (k, \frac{1+k}{2}), (k, \frac{1-k}{2})) = (L(r, 1), \mathcal{F}^-)$  with  $r = k$  odd.



	$\mathcal{O}$	$\mathcal{O}_g$	period spectrum
1.	$S^2(p, q), 2 (p+q)$	$S^2((p+q)/2, (p+q)/2)$	$((p+q)/2)$
1'.	$S^2(p, q), 2 \nmid (p+q)$	$S^2(p+q, p+q)$	$(\overline{p+q})$
2.	$S^2(2, 2, 2n)$	$S^2(2, 2, 2n)$	$(2, 2, 2n)$
3.	$S^2(2, 2, 2n+1)$	$S^2(2, 2, 2n+1)$	$(\overline{2}, 2n+1)$
4.	$S^2(2, 3, 3)$	$S^2(2, 3, 3)$	$(2, \overline{3})$
5.	$S^2(2, 3, 4)$	$S^2(2, 3, 4)$	$(2, 3, 4)$
6.	$S^2(2, 3, 5)$	$S^2(2, 3, 5)$	$(2, 3, 5)$
7.	$D^2(; p, q), 2 (p+q)$	$S^2(2, 2, (p+q)/2)$	$((p+q)/2)$
7'.	$D^2(; p, q), 2 \nmid (p+q)$	$S^2(2, 2, p+q)$	$(2, p+q)$
8.	$\mathbb{RP}^2(2n)$	$S^2(4n, 4n)$	$(4n)$
9.	$D^2(2n+1; )$	$S^2(4n+2, 4n+2)$	$(\overline{2n+1})$
10.	$D^2(2; 2n)$	$S^2(2, 2, 4n)$	$(2, 4n)$
11.	$D^2(; 2, 2, 2n+1)$	$S^2(2, 2, 4n+2)$	$(2, 2n+1)$
12.	$D^2(; 2, 3, 3)$	$S^2(2, 3, 4)$	$(3, 4)$
13.	$\mathbb{RP}^2(2n+1)$	$S^2(2n+1, 2n+1)$	$(2n+1)$
14.	$D^2(2n; )$	$S^2(2n, 2n)$	$(\overline{2n})$
15.	$D^2(2; 2n+1)$	$S^2(2, 2, 2n+1)$	$(2n+1)$
16.	$D^2(; 2, 2, 2n)$	$S^2(2, 2, 2n)$	$(n)$
17.	$D^2(3; 2)$	$S^2(2, 3, 3)$	$(\overline{3})$
18.	$D^2(; 2, 3, 4)$	$S^2(2, 3, 4)$	$(2, 3)$
19.	$D^2(; 2, 3, 5)$	$S^2(2, 3, 5)$	$(3, 5)$

TABLE 1. Orbifolds of prime geodesics  $\mathcal{O}_g$  and geodesic period spectra of 2-orbifolds  $\mathcal{O}$  admitting a Besse metric. A spectrum of  $(\overline{2}, 3)$  means that there are three exceptional prime geodesics that are 2-, 2- and 3-times shorter than the regular prime geodesics. The bar indicates that the geodesics of half the regular period are inverse to each other, i.e. they can be traversed with two different orientations and have the same trajectory on  $\mathcal{O}$ . The other exceptional geodesic of order 3 admits only one orientation (cf. Section 2.7).

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